Strong Convergence of Wong-Zakai Approximations of Reflected SDEs in A Multidimensional General Domain

Tusheng Zhang[1]

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Abstract

In this paper, we obtained the strong convergence of Wong-Zakai approximations of reflected SDEs in a general multidimensional domain giving an affirmative answer to the question posed in [ES].

1 Introduction

Let D be a bounded domain in R^d. Consider the reflected stochastic differential equation (SDE):

$$\begin{cases} dX(t) = \sigma(X(t)) \circ dW(t) + b(X(t))dt + dL(t), \\ X(0) = x_0, \quad X(t) \in \bar{D}, t \geq 0, \\ |L|(t) = \int_0^t I_{\partial D}(X(s))d|L|(s), \end{cases} \quad (1.1)$$

where $W(t), t \geq 0$ is a m-dimensional Brownian motion, $|L|(t)$ stands for the total variation of L on the interval $[0,t]$, $\circ$ indicates a Stratonovich integral.

There is a big amount of literature devoted to the study of reflected SDEs. Let us mention a few of them. Reflected SDEs in a convex domain was first studied by H. Tanaka in [T]. Existence and uniqueness of solutions of reflected SDEs in general domains were established by Lions and Sznitman in [LS] and Saisho in [S]. Existence and uniqueness of solutions of reflected SDEs under more general coefficients than the usual Lipschitz conditions were considered in [MR].

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[1] School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, U.K. Email: tusheng.zhang@manchester.ac.uk

The purpose of this paper is to study Wong-Zakai type approximations of above reflected SDEs. Let W^n be the $n-$dyadic piecewise linear interpolation of W and X^n the solution of the following reflected random ordinary differential equation:

$$\begin{cases} \dot{X}^n(t) = \sigma(X^n(t))\dot{W}^n(t) + b(X^n(t))dt + \dot{L}^n(t), \\ X^n(0) = x_0, \quad X^n(t) \in \bar{D}, t \geq 0, \\ |L^n|(t) = \int_0^t I_{\partial D}(X^n(s))d|L^n|(s). \end{cases} \tag{1.2}$$

We are concerned with the strong convergence of X^n to the solution X. Strong convergence of Wong-Zakai approximations to stochastic differential equations is well known, see e.g. [IW]. However, the convergence of Wong-Zakai approximations to stochastic differential equations with reflection (especially in higher dimensions) is quite tricky because of the constraints on the solution and the appearance of the boundary local time. As far as we are aware of, there are two main papers related to this question. In [P], Petterson established a Wong-Zakai approximations for SDEs with reflection under the assumption that the domain is convex. The convexity is too rigid sometimes for applications. In [ES], Evans and Stroock considered Wong-Zakai approximations for reflected SDEs in general domains (as in [LS]) and proved that X^n converges weakly (in law) to the solution X. In the same paper, the authors also posed the question of whether the strong convergence holds. For some of the interesting applications, we refer the reader to [ES].

The purpose of this paper is to establish the strong convergence (the L^p convergence in $C([0,T],\bar{D})$ of the Wong-Zakai approximations for reflected SDEs in multidimensional general domains, hence giving an affirmative answer to the question in [ES].

The paper is organized as follows. In Section 2, we recall the framework and formulate the main result. The rest of the paper (Section 3) is entirely devoted to the proof of the theorem.

2 Framework and the main result

Let $D \subset R^d$ be a bounded domain with boundary ∂D. For $x \in \partial D$, let $\nu(x) \subset S^{d-1}$ denote a nonempty collection of reflecting directions. Throughout this paper, as in [LS], [ES], we impose the following conditions on the domain.

D.1 $\nu(x) \neq \emptyset$ for every $x \in \partial D$ and there exist a constant $C_0 \geq 0$ such that

$$(x'-x)\cdot\nu+C_0|x-x'|^2 \geq 0 \qquad \text{for all} \quad x' \in D, x \in \partial D \quad \text{and} \quad \nu \in \nu(x).$$

D.2 There exists a function $\phi \in C^2(R^d; R)$ and $\alpha > 0$ such that

$$\nabla\phi(x) \cdot \nu \geq \alpha \qquad \text{for all} \quad x \in \partial D \quad \text{and} \quad \nu \in \nu(x).$$

D.3 There exist $n \geq 1$, $\lambda > 0$, $K > 0$, $a_1, a_2, ..., a_n \in S^{d-1}$, and $x_1, x_2, ..., x_n \in \partial D$ such that $\partial D \subset \cup_{i=1}^n B(x_i, K)$ and $x \in \partial D \cap B(x_i, 2K) \Longrightarrow \nu \cdot a_i \geq \lambda$ for all $\nu \in \nu(x)$.

Convention; Throughout this paper, any function G defined on the positive half line $[0, \infty)$ automatically extends to a function on the whole line by setting $G(s) = G(s \vee 0)$ when necessary.

Let $W(t) = (W_1(t), W_2(t), ..., W_m(t))$, $t \geq 0$ be a m-dimensional Brownian motion on a completed filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Suppose $\sigma = (\sigma_{i,j}) \in C^1(\bar{D}; R^d \otimes R^m)$ such that the derivative σ' is Lipschitz continuous and that $b : \bar{D} \to R^d$ is Lipschitz continuous.

For $n \in N$ and $s \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, set $s_n^- = (\frac{k-1}{2^n}) \vee 0$ and $s_n = \frac{k}{2^n}$. Let W^n be the linear interpolation of W defined by

$$W^n(t) = W(\frac{k-1}{2^n}) + 2^n(t - \frac{k}{2^n})(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) \qquad (2.1)$$

for $t \in [\frac{k}{2^n}, \frac{k+1}{2^n})$, $\quad k = 0, 1, 2, ...$ Note that the above convention applies here. Let $\sigma\sigma' : \bar{D} \to R^d$ be defined as

$$(\sigma\sigma'(y))_i = \sum_{j=1}^m \sum_{k=1}^d \frac{\partial \sigma_{i,j}(y)}{\partial y_k} \sigma_{k,j}(y). \qquad (2.2)$$

With this notation, equation (1.1) becomes

$$X(t) = x_0 + \int_0^t \sigma(X(s))dW(s) + \frac{1}{2}\int_0^t \sigma\sigma'(X(s))ds + \int_0^t b(X(s))ds + L(t) \qquad (2.3)$$

Definition 2.1 *We say that (X, L) is a solution to the reflected SDE (2.3) if (X, L) is a $\bar{D} \times R^d$-valued, adapted continuous process such that*
(i) $L(t), t \geq 0$ is of bounded variation on any finite sub-interval of $[0, \infty)$,
(ii) for $t \geq 0$,

$$X(t) = x_0 + \int_0^t \sigma(X(s))dW(s) + \frac{1}{2}\int_0^t \sigma\sigma'(X(s))ds + \int_0^t b(X(s))ds + L(t)$$

almost surely,
(iii)

$$|L|(t) = \int_0^t I_{\partial D}(X(s))d|L|(s), \qquad L(t) = \int_0^t \nu(X(s))d|L|(s),$$

where $|L|(t)$ stands for the total variation of L on the interval $[0, t]$, the last equality means that $\frac{DL(t)}{d|L|(t)} \in \nu(X(t))$.

The solution (X^n, L^n) to the reflected random ordinary differential equation (1.2) is defined accordingly.

Under the above assumptions, the existence and uniqueness of X^n, X are well known now, see, for example, [LS]. Here is the main result.

Theorem 2.2 *Let X^n, X be the solutions to reflected stochastic equations (1.1) and (1.2). It holds that for any $p > 0$ and $T > 0$,*

$$\lim_{n \to \infty} E[\sup_{0 \le t \le T} |X^n(t) - X(t)|^p] = 0. \tag{2.4}$$

In next section, C will denote a generic constant which is usually different from line to line.

3 The roof of the main result

The rest of the paper is devoted to the proof of Theorem 2.2. First of all we recall the following estimate from [ES].

Lemma 3.1 *Let $p \ge 2$, $T > 0$. Then there exists a constant $C_1(T, p)$ independent of n such that*

$$E[|X^n(t) - X^n(s)|^p] \le C_1(T, p)|t - s|^{\frac{p}{2}}, \tag{3.1}$$

for $0 \le s, t \le T$.

Applying the proof in [ES] to $X(t), t \ge 0$ one also has

Lemma 3.2 *Let $p \ge 2$, $T > 0$. Then there exists a constant $C_2(T, p)$ such that*

$$E[|X(t) - X(s)|^p] \le C_2(T, p)|t - s|^{\frac{p}{2}}, \tag{3.2}$$

for $0 \le s, t \le T$.

Due to (3.1), (3.2) above, to prove Theorem 2.2, it is sufficient to show that for any fixed $t > 0$

$$\lim_{n \to \infty} E[|X^n(t) - X(t)|^2] = 0. \tag{3.3}$$

Indeed, it follows from (3.1), (3.2) and Garsia, Rodemich and Rumsey's lemma (See Theorem 1.1 in [W]) that for a fixed positive number $\alpha_0 < \frac{1}{2}$, there exist random variables $K_n(\omega), K(\omega)$ such that

$$|X^n(t) - X^n(s)| \le K_n(\omega)|t - s|^{\alpha_0}, \quad s, t \in [0, T], \tag{3.4}$$

and

$$|X(t) - X(s)| \le K(\omega)|t - s|^{\alpha_0}, \quad s, t \in [0, T]. \tag{3.5}$$

Furthermore, because the constant $C_1(T, p)$ in (3.1) is independent of n, K_n, K can be chosen to satisfy

$$\sup_n E[K_n^p] < \infty, \quad E[K^p] < \infty, \tag{3.6}$$

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for any $p > 0$. Since X^n, X live on the bounded domain $\bar{D}$, to show (2.4) it is sufficient to prove that X^n converges to X in probability. For $\varepsilon > 0$, choose $t_i \in [0, T], i = 1, ..., N_\varepsilon$ such that $[0, T] \subset \cup_i B(t_i, \varepsilon)$. Given $\delta > 0$, for $M > 0$, we have

$$P(\sup_{0 \le t \le T} |X^n(t) - X(t)| > \delta)$$

$$\le \quad P(\sup_i \sup_{t \in B(t_i, \varepsilon)} |X^n(t) - X^n(t_i)| > \frac{\delta}{3}, |K_n| \le M) + P(|K_n| > M)$$

$$+ \quad P(\sup_i \sup_{t \in B(t_i, \varepsilon)} |X(t) - X(t_i)| > \frac{\delta}{3}, |K| \le M) + P(|K| > M)$$

$$+ \quad \sum_{i=1}^{N_\varepsilon} P(|X^n(t_i) - X(t_i)| > \frac{\delta}{3}). \tag{3.7}$$

Now, for any given $\eta > 0$, by (3.6) we first choose M sufficiently large so that $P(|K_n| > M) \le \frac{\eta}{4}$, $P(|K| > M) \le \frac{\eta}{4}$ for all n. For such a constant M, because of (3.4) and (3.5) we can select $\varepsilon > 0$ sufficiently small so that

$$P(\sup_i \sup_{t \in B(t_i, \varepsilon)} |X^n(t) - X^n(t_i)| > \frac{\delta}{3}, |K_n| \le M) = 0,$$

and

$$P(\sup_i \sup_{t \in B(t_i, \varepsilon)} |X(t) - X(t_i)| > \frac{\delta}{3}, |K| \le M) = 0.$$

When ε is fixed, it follows from (3.3) that there exists $N > 0$ such that for $n \ge N$,

$$\sum_{i=1}^{N_\varepsilon} P(|X^n(t_i) - X(t_i)| > \frac{\delta}{3}) < \frac{\eta}{4}$$

Putting the above arguments together we prove that X^n converges to X in probability.

So we remain to prove (3.3). Again because of (3.1), (3.2) we may assume that t is a dyadic number, i.e., $t = \frac{k_0}{2^{n_0}}$ for some positive integers k_0, n_0 and we may also assume $n \ge n_0$.

Let $f(y_1, y_2, y_3) = exp(r(y_1 + y_2))y_3$. Recall ϕ is the function specified in (D.2). To simplify the exposure, we introduce the following notation:

$$y_1(t) := \phi(X(t)), y_2^n(t) := \phi(X^n(t)), y_3^n(t) := |X^n(t) - X(t)|^2.$$

$$f_n(t) := f(y_1(t), y_2^n(t), y_3^n(t)), g_n(t) := exp(ry_1(t) + ry_2^n(t)).$$

Since X^n, X take values in the bounded domain $\bar{D}$, we have

$$c_1|X^n(t) - X(t)|^2 \le f_n(t) \le c_2|X^n(t) - X(t)|^2, \tag{3.8}$$

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where c_1, c_2 are positive constants independent of n. Thus the proof of (3.3) reduces to show

$$\lim_{n\to\infty} E[f_n(t)] = 0. \tag{3.9}$$

By Ito's formula, we have

$$
\begin{aligned}
&f_n(t) \\
=~& r \int_0^t f_n(s) < \nabla\phi(X(s)), \sigma(X(s))dW(s) > + r \int_0^t f_n(s) < \nabla\phi(X(s)), b(X(s) > ds \\
+~& \frac{1}{2} r \int_0^t f_n(s) tr(\phi''(\sigma\sigma^*)(X(s))) ds + \frac{1}{2} r \int_0^t f_n(s) < \nabla\phi(X(s), \sigma\sigma'(X(s)) > ds \\
+~& r \int_0^t f_n(s) < \nabla\phi(X(s)), \nu(X(s)) > d|L|(s) + r \int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+~& r \int_0^t f_n(s) < \nabla\phi(X^n(s)), b(X^n(s)) > ds + r \int_0^t f_n(s) < \nabla\phi(X^n(s)), \nu(X^n(s)) > d|L^n|(s) \\
+~& 2 \int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
-~& 2 \int_0^t g_n(s) < X^n(s) - X(s), \sigma(X(s))dW(s) > \\
+~& 2 \int_0^t g_n(s) < X^n(s) - X(s), b(X^n(s)) - b(X(s)) > ds \\
-~& \int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X(s)) > ds \\
+~& 2 \int_0^t g_n(s) < X^n(s) - X(s), \nu(X^n(s))d|L^n|(s) - \nu(X(s))d|L|(s) > \\
+~& \int_0^t g_n(s) tr(\sigma\sigma^*(X(s))) ds + \frac{1}{2} r^2 \int_0^t f_n(s) |\sigma^*\nabla\phi|^2(X(s)) ds \\
-~& 2r \int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X(s)) > ds. \tag{3.10}
\end{aligned}
$$

$$
\begin{aligned}
&g_n(t) \\
=~& exp(2r\phi(x_0)) + r \int_0^t g_n(s) < \nabla\phi(X(s)), \sigma(X(s))dW(s) > \\
+~& r \int_0^t g_n(s) < \nabla\phi(X(s)), b(X(s)) > ds \\
+~& \frac{1}{2} r \int_0^t g_n(s) tr(\phi''(\sigma\sigma^*)(X(s))) ds + \frac{1}{2} r \int_0^t g_n(s) < \nabla\phi(X(s), \sigma\sigma'(X(s)) > ds \\
+~& r \int_0^t g_n(s) < \nabla\phi(X(s)), \nu(X(s)) > d|L|(s) \\
+~& r \int_0^t g_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+~& r \int_0^t g_n(s) < \nabla\phi(X^n(s)), b(X^n(s)) > ds + r \int_0^t g_n(s) < \nabla\phi(X^n(s)), \nu(X^n(s)) > d|L^n|(s) \\
+~& \frac{1}{2} r^2 \int_0^t g_n(s) |\sigma^*\nabla\phi|^2(X(s)) ds \tag{3.11}
\end{aligned}
$$

To bound $E[f_n(t)]$, the crucial step is to get proper estimates for the terms

$$rE[\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >],$$

and

$$rE[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >].$$

This will be done in the following two lemmas.

Lemma 3.3 *It holds that*

$$
\begin{aligned}
&rE[\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >] \\
\leq\ & C(\frac{1}{2^n})^{\frac{1}{2}} + r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*\nabla\phi(X^n(s)) > ds] \\
& +\frac{1}{2}r^2 E[\int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds] \\
& +r\int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds \\
& +\frac{1}{2}r\int_0^t f_n(s)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i)))_i(X^n(s))ds \\
& -2r\int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds (3.12)
\end{aligned}
$$

Proof. Set

$$A = r\int_0^t f_n(s) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > .$$

Write

$$
\begin{aligned}
A &= r\int_0^t f_n(s_n^-) < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) > \\
&+ r\int_0^t (f_n(s) - f_n(s_n^-)) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
&+ r\int_0^t f_n(s_n^-) < \sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(s_n^-)), dW^n(s) > \\
&:= A_1 + A_2 + A_3. \quad\quad (3.13)
\end{aligned}
$$

As a stochastic integral, it is easy to see that $E[A_1] = 0$. In view of (3.10), we further write A_2 as

$$
\begin{aligned}
&A_2 \\
=\ & r^2\int_0^t(\int_{s_n^-}^s f_n(u) < \nabla\phi(X(u)), \sigma(X(u))dW(u) >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+\ & r^2\int_0^t(\int_{s_n^-}^s f_n(u) < \nabla\phi(X(u)), b(X(u))du >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+\ & \frac{1}{2}r^2\int_0^t(\int_{s_n^-}^s f_n(u)tr(\phi''(\sigma\sigma^*)(X(u))du) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >
\end{aligned}
$$

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$$+ \quad \frac{1}{2}r^2 \int_0^t (\int_{s_n^-}^s f_n(u) < \nabla\phi(X(u)), (\sigma\sigma')(X(u)) > du) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad r^2 \int_0^t (\int_{s_n^-}^s f_n(u) < \nabla\phi(X(u)), \nu(X(u)) > d|L|(u)) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad r^2 \int_0^t (\int_{s_n^-}^s f_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u))dW^n(u) >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad r^2 \int_0^t (\int_{s_n^-}^s f_n(u) < \nabla\phi(X^n(u)), b(X^n(u))du >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad r^2 \int_0^t (\int_{s_n^-}^s f_n(u) < \nabla\phi(X^n(u)), \nu(X^n(u)) > d|L^n|(u)) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad 2r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), \sigma(X^n(u))dW^n(u) >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), \sigma(X(u))dW(u) >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad 2r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), b(X^n(u)) - b(X(u)) > du) < \nabla\phi(X^n(s)),$$
$$\sigma(X^n(s))dW^n(s) >$$

$$- \quad r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), \sigma\sigma'(X(u)) > du) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad 2r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), \nu(X^n(u)) > d|L^n|(u))$$
$$\times < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2r \int_0^t (\int_{s_n^-}^s g_n(u) < X^n(u) - X(u), \nu(X(u)) > d|L|(u)) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad r \int_0^t (\int_{s_n^-}^s g_n(u) tr(\sigma\sigma^*(X(u)))du) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad \frac{1}{2}r^3 \int_0^t (\int_{s_n^-}^s f_n(u)|\sigma^*\nabla\phi|^2(X(u)))du) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2r^2 \int_0^t (\int_{s_n^-}^s g_n(u) < \sigma^*(X(u))(X^n(u) - X(u)), \sigma^*\nabla\phi(X(u)) > du)$$
$$\times < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) >$$

$$:= \quad \sum_{i=1}^{17} A_{2i} \tag{3.14}$$

We will bound each of the terms. Since $\nabla\phi$, b, σ are bounded on $\bar{D}$, we have

$$E[|A_{22}|] \leq C \int_0^t (s - s_n^-)E[|\dot{W}^n(s)|]ds$$

$$\leq C\frac{1}{2^n} \int_0^t (2^n)^{\frac{1}{2}}ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.15}$$

Similarly, it holds that

$$E[|A_{2i}|] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \quad i = 3, 4, 7, 11, 12, 15, 16, 17. \tag{3.16}$$

To bound A_{21}, we write it as

$$
\begin{aligned}
& A_{21} \\
=\ & r^2 \int_0^t (\int_{s_n^-}^s [f_n(u) < \nabla\phi(X(u)), \sigma(X(u)dW(u) > -f_n(s_n^-) < \nabla\phi(X(s_n^-)), \\
& \quad \sigma(X(s_n^-))dW(u) >]) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+\ & r^2 \int_0^t (\int_{s_n^-}^s f_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))dW(u) >) \\
& \quad \times [< \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > - < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) >] \\
+\ & r^2 \int_0^t f_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))(W(s) - W(s_n^-)) > \\
& \quad \times < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) > \\
:=\ & A_{21,1} + A_{21,2} + A_{21,3}. \qquad (3.17)
\end{aligned}
$$

By Ito isometry and Hölder's inequality,

$$
\begin{aligned}
& E[A_{21,1}] \\
\leq\ & C \int_0^t (E[\int_{s_n^-}^s |f_n(u)\sigma^*\nabla\phi(X(u)) - f_n(s_n^-)\sigma^*\nabla\phi(X(s_n^-))|^2 du])^{\frac{1}{2}} (E[|\dot{W}^n|^2(s)])^{\frac{1}{2}} ds \\
\leq\ & C \int_0^t (2^n)^{\frac{1}{2}} (E[\int_{s_n^-}^s |f_n(u)\sigma^*\nabla\phi(X(u)) - f_n(s_n^-)\sigma^*\nabla\phi(X(s_n^-))|^2 du])^{\frac{1}{2}} ds \\
\leq\ & C \int_0^t (2^n)^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad (3.18)
\end{aligned}
$$

where (3.1), (3.2) have been used. For the term $A_{21,2}$, we have

$$
\begin{aligned}
& E[A_{21,2}] \\
\leq\ & C \int_0^t (E[|W(s) - W(s_n^-)|^3])^{\frac{1}{3}} (E[|\sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(s_n^-))|^3])^{\frac{1}{3}} \\
& \quad \times (E[|\dot{W}^n|^3(s)])^{\frac{1}{3}} ds \\
\leq\ & C \int_0^t (2^n)^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (3.19)
\end{aligned}
$$

where (3.1) has been used. Now,

$$
\begin{aligned}
& A_{21,3} \\
=\ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f_n(\frac{k-1}{2^n}) < \nabla\phi(X(\frac{k-1}{2^n})), \sigma(X(\frac{k-1}{2^n})(W(s) - W(\frac{k}{2^n})) > \\
& \quad \times < \nabla\phi(X^n(\frac{k-1}{2^n})), \sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) > ds \\
+\ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) > \\
& \quad \times < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > ds \\
:=\ & A_{21,31} + A_{21,32}. \qquad (3.20)
\end{aligned}
$$

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Conditioning on $\mathcal{F}_{\frac{k}{2^n}}$, it is easy to see that $E[A_{21,31}] = 0$. Moreover,

$$
\begin{aligned}
& A_{21,32} \\
=\; & r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^{m} (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*\nabla\phi)_i(X^n(\frac{k-1}{2^n})) \\
& \times(|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}) \\
+\; & r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i\neq j} (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*\nabla\phi)_j(X^n(\frac{k-1}{2^n})) \\
& \times(W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})) \\
+\; & r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^{m} (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*\nabla\phi)_i(X^n(\frac{k-1}{2^n}))(\frac{1}{2^n}) \\
:=\; & A_{21,321} + A_{21,322} + A_{21,323}. \quad\quad\quad (3.21)
\end{aligned}
$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ and using the independence of W_i, W_j for $i \neq j$, we find that $E[A_{21,321}] = 0$ and $E[A_{21,322}] = 0$. On the other hand,

$$
\begin{aligned}
& E[A_{21,323}] \\
=\; & r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*\nabla\phi(X^n(s)) > ds] \\
+\; & r^2 E[\int_0^t \{f_n(s_n^-) < \sigma^*\nabla\phi(X(s_n^-)), \sigma^*\nabla\phi(X^n(s_n^-)) > \\
& - f_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*\nabla\phi(X^n(s)) >\}ds] \\
\leq\; & r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*\nabla\phi(X^n(s)) > ds] + C(\frac{1}{2^n})^{\frac{1}{2}} (3.22)
\end{aligned}
$$

where (3.1), (3.2) again have been used. Putting together (3.17)—(3.22) we arrive at

$$
E[A_{21}] \leq CE[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad\quad (3.23)
$$

The term A_{25} can be bounded as follows.

$$
\begin{aligned}
& E[A_{25}] \\
\leq\; & CE[\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{s} d|L|(u))2^n|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|ds] \\
\leq\; & CE[\sum_k (|L|(\frac{k}{2^n}) - |L|(\frac{k-1}{2^n}))|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|] \\
\leq\; & 2CE[|L|(t) \sup_{|u-v|\leq\frac{1}{2^n}} (|W(u) - W(v)|)] \\
\leq\; & 2C(E[|L|^2(t)])^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad\quad (3.24)
\end{aligned}
$$

To control the term A_{26}, we write it as

$$
A_{26}
$$

$$
\begin{aligned}
= \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u))dW^n(u) >) \\
& \times < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+ \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k}{2^n}}^{s} f_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u))dW^n(u) >) \\
& \times < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
= \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} < f_n(u)\sigma^*\nabla\phi(X^n(u)) - f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), \\
& dW^n(u) >) < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
+ \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), dW^n(u) >) \\
& \times < \sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), dW^n(s) > \\
+ \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} f_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), dW^n(u) >) \\
& \times < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), dW^n(s) > \\
+ \ & r^2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k}{2^n}}^{s} f_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u))dW^n(u) >) \\
& \times < \nabla\phi(X^n(s)), \sigma(X^n(s))dW^n(s) > \\
:= \ & A_{26,1} + A_{26,2} + A_{26,3} + A_{26,4} \quad\quad\quad\quad\quad\quad\quad (3.25)
\end{aligned}
$$

The first term on the right can be bounded as follows:

$$
\begin{aligned}
& E[A_{26,1}] \\
\leq \ & C \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du E[|\dot{W}^n(u)||\dot{W}^n(s)| \\
& \times |f_n(u)\sigma^*\nabla\phi(X^n(u)) - f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|] \\
\leq \ & C \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du (2^n)^{\frac{1}{2}} (2^n)^{\frac{1}{2}} \\
& \times (E[|f_n(u)\sigma^*\nabla\phi(X^n(u)) - f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|^3])^{\frac{1}{3}} \\
\leq \ & C \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du (2^n)^{\frac{1}{2}} (2^n)^{\frac{1}{2}} (\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad (3.26)
\end{aligned}
$$

The second term has the following upper bound.

$$
\begin{aligned}
& E[A_{26,2}] \\
\leq \ & C \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}\vee 0}^{\frac{k}{2^n}} du E[|\dot{W}^n(u)||\dot{W}^n(s)||\sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|]
\end{aligned}
$$

$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}\vee 0}^{\frac{k}{2^n}} du (2^n)^{\frac{1}{2}}(2^n)^{\frac{1}{2}} (E[|\sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|^3])^{\frac{1}{3}}$$

$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}\vee 0}^{\frac{k}{2^n}} du (2^n)^{\frac{1}{2}}(2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.27}$$

Note that

$$A_{26,3}$$
$$= r^2\sum_k (2^n)^2 \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du f_n(\frac{k-1}{2^n}) \sum_{i,j=1}^{m}(\sigma^*\nabla\phi)_i(X^n(\frac{k-1}{2^n})(\sigma^*\nabla\phi)_j(X^n(\frac{k-1}{2^n})))$$
$$\times (W_i(\frac{k-1}{2^n}) - W_i(\frac{k-2}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})). \tag{3.28}$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ we see that $E[A_{26,3}] = 0$. On the other hand, the term $A_{26,4}$ can be further split as follows.

$$A_{26,4}$$
$$= r^2\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k}{2^n}}^{s} < f_n(u)\sigma^*\nabla\phi(X^n(u)) - f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), \dot{W}^n(u) > du)$$
$$\times < \sigma^*\nabla\phi(X^n(s)), \dot{W}^n(s) > ds$$
$$+ r^2\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k}{2^n}}^{s} < f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), \dot{W}^n(u) > du)$$
$$\times < \sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), \dot{W}^n(s) > ds$$
$$+ r^2(2^n)^2\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^{s} du f_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) >^2$$
$$:= A_{26,41} + A_{26,42} + A_{26,43} \tag{3.29}$$

By the Lipschitz continuity of the coefficients and (3.1) and (3.2) we have

$$E[A_{26,41}]$$
$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^{s} du E[||\dot{W}^n(u)||\dot{W}^n(s)|$$
$$\times |f_n(u)\sigma^*\nabla\phi(X^n(u)) - f_n(\frac{k-1}{2^n})\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|]$$
$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} du (2^n)^{\frac{1}{2}}(2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.30}$$

and

$$E[A_{26,42}]$$
$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^{s} du E[||\dot{W}^n(u)||\dot{W}^n(s)| \cdot |\sigma^*\nabla\phi(X^n(s)) - \sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|]$$
$$\leq C\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} du (2^n)^{\frac{1}{2}}(2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.31}$$

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Furthermore,

$$A_{26,43}$$

$$= \frac{1}{2}r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i \neq j}^m (\sigma^*\nabla\phi)_i(X^n(\frac{k-1}{2^n}))(\sigma^*\nabla\phi)_j(X^n(\frac{k-1}{2^n}))$$

$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))$$

$$+ \frac{1}{2}r^2 \sum_k f_n(\frac{k-1}{2^n}) \sum_{i=1}^m (\sigma^*\nabla\phi)_i^2(X^n(\frac{k-1}{2^n}))(|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n})$$

$$+ \frac{1}{2}r^2 \int_0^t [f_n(s_n^-)|\sigma^*\nabla\phi|^2(X^n(s_n^-)) - f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))]ds$$

$$+ \frac{1}{2}r^2 \int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds. \tag{3.32}$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ and using the independence of W_i, W_j for $i \neq j$, it is easy to see that the expectation of the first two terms on the right side are zero. By (3.1), the expectation of the third term is bounded by $C(\frac{1}{2^n})^{\frac{1}{2}}$. Thus we conclude that

$$E[A_{26,43}]$$
$$\leq C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2}r^2 E[\int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds]. \tag{3.33}$$

Combining (3.29)—(3.33), we find that

$$E[A_{26,4}]$$
$$\leq C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2}r^2 E[\int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds]. \tag{3.34}$$

Putting together (3.25),(3.26),(3.27), (3.28) and (3.34) yields

$$E[A_{26}]$$
$$\leq C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2}r^2 E[\int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds]. \tag{3.35}$$

The term A_{28} admits a similar bound as A_{25}:

$$E[A_{28}]$$
$$\leq CE[|L^n|(t) \sup_{|u-v| \leq \frac{1}{2^n}} (|W(u) - W(v)|)]$$
$$\leq 2C[\sup_n (E[|L^n|^2(t)])^{\frac{1}{2}}](\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.36}$$

Now let us turn to A_{29}. We have

$$A_{29}$$

$$= 2r \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} g_n(u) < X^n(u) - X(u), \sigma(X^n(u))dW^n(u) >)$$

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$$\times <\nabla\phi(X^n(s)),\sigma(X^n(s))dW^n(s)>$$

$$+\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}(\int_{\frac{k}{2^n}}^{s}g_n(u)<X^n(u)-X(u),\sigma(X^n(u))dW^n(u)>)$$

$$\times <\nabla\phi(X^n(s)),\sigma(X^n(s))dW^n(s)>$$

$$=\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}(\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}}g_n(u)<X^n(u)-X(u),\sigma(X^n(u))dW^n(u)>)$$

$$\times <\sigma^*\nabla\phi(X^n(s))-\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),dW^n(s)>$$

$$+\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}(\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}}<g_n(u)\sigma^*(X^n(u))(X^n(u)-X(u))$$

$$-g_n(\frac{k-1}{2^n})\sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n})-X(\frac{k-1}{2^n})),dW^n(u)>)$$

$$\times <\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),dW^n(s)>$$

$$+\quad 2r\sum_k<g_n(\frac{k-1}{2^n})\sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n})-X(\frac{k-1}{2^n})),W(\frac{k-1}{2^n})-W(\frac{k-2}{2^n})>$$

$$\times <\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),W(\frac{k}{2^n})-W(\frac{k-1}{2^n})>$$

$$+\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}(\int_{\frac{k}{2^n}}^{s}g_n(u)<X^n(u)-X(u),\sigma(X^n(u))dW^n(u)>)$$

$$\times <\sigma^*\nabla\phi(X^n(s))-\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),dW^n(s)>$$

$$+\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}(\int_{\frac{k}{2^n}}^{s}<g_n(u)\sigma^*(X^n(u))(X^n(u)-X(u))-g_n(\frac{k-1}{2^n})\sigma^*(X^n(\frac{k-1}{2^n}))$$

$$\times(X^n(\frac{k-1}{2^n})-X(\frac{k-1}{2^n})),dW^n(u)>)<\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),dW^n(s)>$$

$$+\quad r\sum_k<g_n(\frac{k-1}{2^n})\sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n})-X(\frac{k-1}{2^n})),W(\frac{k}{2^n})-W(\frac{k-1}{2^n})>$$

$$\times <\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n})),W(\frac{k}{2^n})-W(\frac{k-1}{2^n})>$$

$$:=\quad \sum_{i=1}^{6}A_{29,i} \tag{3.37}$$

The first and the second term on the right have the following bounds.

$$E[A_{29,1}]$$

$$\leq\quad 2r\sum_k\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}}ds\int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}}duE[||\dot{W}^n(u)||\dot{W}^n(s)||\sigma^*\nabla\phi(X^n(s))-\sigma^*\nabla\phi(X^n(\frac{k-1}{2^n}))|]$$

$$\leq\quad C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.38}$$

$$E[A_{29,2}]$$

$$\leq \quad 2r \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du E[|\dot{W}^n(u)||\dot{W}^n(s)|$$

$$\times |g_n(u)\sigma^*(X^n(u))(X^n(u) - X(u)) - g_n(\frac{k-1}{2^n})\sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}))|]$$

$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}}, \tag{3.39}$$

here (3.1), (3.2) have been used again. By a similar reason, it also holds that

$$E[A_{29,4}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.40}$$

$$E[A_{29,5}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.41}$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$, we have that $E[A_{29,3}] = 0$. Note that

$$A_{29,6}$$

$$= \quad r \sum_k \sum_{i \neq j} [g_n(\frac{k-1}{2^n})(\sigma^*(X^n(\frac{k-1}{2^n})))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n})))_i(\sigma^*\nabla\phi)_j(X^n(\frac{k-1}{2^n}))$$

$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))]$$

$$+ \quad r \sum_k \sum_{i=1}^m [g_n(\frac{k-1}{2^n})(\sigma^*(X^n(\frac{k-1}{2^n})))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n})))_i(\sigma^*\nabla\phi)_i(X^n(\frac{k-1}{2^n}))$$

$$\times \{|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}\}$$

$$+ \quad r \int_0^t \{< g_n(s_n^-)\sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)), \sigma^*\nabla\phi(X^n(s_n^-)) >$$

$$- < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) >\}ds$$

$$+ \quad r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds$$

$$:= \quad A_{29,61} + A_{29,62} + A_{29,63} + A_{29,64} \tag{3.42}$$

Again by conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ and the independence,

$$E[A_{29,61}] = 0, \quad E[A_{29,62}] = 0.$$

By virtue of (3.1) and (3.2),

$$E[A_{29,63}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.43}$$

It follows from (3.37)—(3.43) that

$$E[A_{29}]$$

$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}} + r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds. \tag{3.44}$$

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Applying the same arguments to A_{210} in (3.14), we get

$$E[A_{210}]$$
$$\leq C(\frac{1}{2^n})^{\frac{1}{2}} - 2r \int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds.$$
$$(3.45)$$

As for the term A_{213} in (3.14), we have

$$E[A_{213}]$$
$$\leq CE[\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^s d|L^n|_u)(2^n|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|ds$$
$$\leq CE[\sum_k (|L^n|_{\frac{k}{2^n}} - |L^n|_{\frac{k-1}{2^n}})|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|$$
$$\leq 2CE[|L^n|_t \sup_{|u-v|\leq\frac{1}{2^n}} (|W(u) - W(v)|)]$$
$$\leq 2C(E[|L^n|_t^2])^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}.$$
$$(3.46)$$

A similar argument leads to

$$E[A_{214}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}.$$
$$(3.47)$$

Collecting the estimates (3.14)—(3.47) we get that

$$E[A_2]$$
$$\leq C(\frac{1}{2^n})^{\frac{1}{2}} + r^2 E[\int_0^t f_n(s) < \sigma^*\nabla\phi(X^n(s)), \sigma^*\nabla\phi(X(s)) > ds]$$
$$+ \frac{1}{2}r^2 E[\int_0^t f_n(s)|\sigma^*\nabla\phi|^2(X^n(s))ds$$
$$+ r \int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds$$
$$- 2r \int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds.$$
$$(3.48)$$

Now we turn to A_3. By the chain rule, we have

$$A_3$$
$$= r \int_0^t f_n(s_n^-) \sum_{i=1}^m [(\sigma^*\nabla\phi)_i(X^n(s)) - (\sigma^*\nabla\phi)_i(X^n(s_n^-))]dW_i^n(s)$$
$$= r \int_0^t f_n(s_n^-) \sum_{i=1}^m \int_{s_n^-}^s [< \nabla(\sigma^*\nabla\phi)_i(X^n(u)) - \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)),$$
$$\sigma(X^n(u))dW^n(u) >]dW_i^n(s)$$
$$+ r \int_0^t f_n(s_n^-) \sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)), (\sigma(X^n(u))$$

$$-\sigma(X^n(s_n^-)))dW^n(u) > dW_i^n(s)$$

$$+ \quad r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(u) > dW_i^n(s)$$

$$+ \quad r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(u)), \nu(X^n(u))d|L^n|(u) > dW_i^n(s)$$

$$+ \quad r\int_0^t f_n(s_n^-)\sum_{i=1}^m \int_{s_n^-}^s < \nabla(\sigma^*\nabla\phi)_i(X^n(u)), b(X^n(u))du > dW_i^n(s)$$

$$:= \quad A_{31} + A_{32} + A_{33} + A_{34} + A_{35} \tag{3.49}$$

Similar to the estimates for A_{214}, A_{22} and the term $A_{21,2}$, it can be shown that

$$E[A_{3i}] \le C(\frac{1}{2^n})^{\frac{1}{2}}, \quad i = 1, 2, 4, 5. \tag{3.50}$$

Now,

$$A_{33}$$
$$= \quad r\sum_k (2^n)^2 \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} du f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n}))$$
$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k-1}{2^n}) - W_j(\frac{k-2}{2^n}))$$
$$+ \quad r\sum_k (2^n)^2 \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds \int_{\frac{k}{2^n}}^s du f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n}))$$
$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))$$
$$= \quad r\sum_k f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n}))$$
$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k-1}{2^n}) - W_j(\frac{k-2}{2^n}))$$
$$+ \quad \frac{1}{2}r\sum_k f_n(\frac{k-1}{2^n})\sum_{i=1}^m\sum_{j=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n}))$$
$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))$$
$$:= \quad A_{331} + A_{332} \tag{3.51}$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$, it is easy to see $E[A_{331}] = 0$. For the second term we have

$$A_{332}$$
$$= \quad \frac{1}{2}r\sum_k f_n(\frac{k-1}{2^n})\sum_{i\ne j}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_j(X^n(\frac{k-1}{2^n}))$$
$$\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))$$

$$+ \quad \frac{1}{2}r\sum_k f_n(\frac{k-1}{2^n})\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(\frac{k-1}{2^n}))\{|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}\}$$

$$+ \quad \frac{1}{2}r\int_0^t \{f_n(s_n^-)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(s_n^-)) - f_n(s)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(s))\}ds$$

$$+ \quad \frac{1}{2}r\int_0^t f_n(s)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(s))ds$$

$$:= \quad A_{3321} + A_{3322} + A_{3323} + A_{3324} \tag{3.52}$$

Using the martingale property and the independence of W_i, W_j for $i \neq j$, we find that $E[A_{3321}] = 0$ and $E[A_{3322}] = 0$. In view of (3.1) and (3.2), we have $E[A_{3323}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}$. Thus, we deduce from (3.51), (3.52) that

$$E[A_{33}]$$
$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2}r\int_0^t f_n(s)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(s))ds. \tag{3.53}$$

Finally it follows from (3.49), (3.50), (3.50) that

$$E[A_3]$$
$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}} + \frac{1}{2}r\int_0^t f_n(s)\sum_{i=1}^m (\sigma^*(\nabla(\sigma^*\nabla\phi)_i))_i(X^n(s))ds. \tag{3.54}$$

Combining (3.48) with (3.54), we complete the proof of Lemma.

Lemma 3.4 *We have*

$$rE[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >]$$

$$\leq \quad rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X^n(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$

$$+ \quad 2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$

$$+ \quad E[\int_0^t g_n(s)\sum_{i=1}^d\sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds]$$

$$+ \quad E[\int_0^t g_n(s)\sum_{i=1}^d (X_i^n(s) - X_i(s))\sum_{j=1}^m (\sigma^*\nabla\sigma_{ij})_j(X^n(s))\}ds]$$

$$- \quad 2E[\int_0^t g_n(s)\sum_{i=1}^d\sum_{j=1}^m \sigma_{ij}(X(s)\sigma_{ij}(X^n(s))ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.55}$$

Proof. Set

$$B = 2\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > .$$

and write

$$
\begin{aligned}
B &= 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), \sigma(X^n(s_n^-))dW^n(s) > \\
&+ 2\int_0^t (g_n(s) - g_n(s_n^-)) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2\int_0^t g_n(s_n^-) < (X^n(s) - X(s)) - (X^n(s_n^-) - X(s_n^-)), \sigma(X^n(s))dW^n(s) > \\
&+ 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), (\sigma(X^n(s)) - \sigma(X^n(s_n^-)))dW^n(s) > \\
&:= B_1 + B_2 + B_3 + B_4. \tag{3.56}
\end{aligned}
$$

As a stochastic integral against Brownian motion, it is easily seen that $E[B_1] = 0$. In view of (3.11),

$$
\begin{aligned}
&B_2 \\
&= 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X(u)), \sigma(X(u))dW(u) >) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X(u)), b(X(u)du >) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ r\int_0^t (\int_{s_n^-}^s g_n(u)tr(\phi''(\sigma\sigma^*)(X(u))du) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X(u)), (\sigma\sigma')(X(u)) > du) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X(u)), \nu(X(u)) > d|L|(u)) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u))dW^n(u) >) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X^n(u)), b(X^n(u))du >) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ 2r\int_0^t (\int_{s_n^-}^s g_n(u) < \nabla\phi(X^n(u)), \nu(X^n(u)) > d|L^n|(u)) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&+ r^2\int_0^t (\int_{s_n^-}^s g_n(u)|\sigma^*\nabla\phi|^2(X(u))du) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) > \\
&:= \sum_{i=1}^9 B_{2i} \tag{3.57}
\end{aligned}
$$

We will closely study each of the terms on the right side. Since $\nabla\phi$, b, σ are bounded on $\tilde{D}$, we have

$$
\begin{aligned}
E[|B_{22}|] &\leq C\int_0^t (s - s_n^-)E[|\dot{W}^n(s)|]ds \\
&\leq C\frac{1}{2^n}\int_0^t (2^n)^{\frac{1}{2}}ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.58}
\end{aligned}
$$

Similar arguments lead to

$$
E[|B_{2i}|] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \quad i = 3, 4, 7, 9. \tag{3.59}
$$

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Regarding B_{25}, we have

$$
\begin{aligned}
&E[B_{25}]\\
\leq\quad & CE[\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (\int_{\frac{k-1}{2^n}}^s d|L|(u))2^n |W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|ds]\\
\leq\quad & CE[\sum_k (|L|(\frac{k+1}{2^n}) - |L|(\frac{k-1}{2^n}))|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|]\\
\leq\quad & 2CE[|L|(t) \sup_{|u-v|\leq \frac{1}{2^n}} (|W(u) - W(v)|)]\\
\leq\quad & 2C(E[|L|^2(t)])^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad (3.60)
\end{aligned}
$$

Similarly,

$$
E[B_{28}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad (3.61)
$$

Now,

$$
\begin{aligned}
&B_{21}\\
=\quad & 2r\int_0^t (\int_{s_n^-}^s [g_n(u) < \nabla\phi(X(u)), \sigma(X(u)dW(u) > -g_n(s_n^-) < \nabla\phi(X(s_n^-)),\\
&\quad \sigma(X(s_n^-))dW(u) >]) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >\\
+\quad & 2r\int_0^t (\int_{s_n^-}^s g_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))dW(u) >)\\
&\quad \times [< \sigma^*(X^n(s))(X^n(s) - X(s)) - \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)), dW^n(s) >]\\
+\quad & 2r\int_0^t g_n(s_n^-) < \nabla\phi(X(s_n^-)), \sigma(X(s_n^-))(W(s) - W(s_n^-)) >\\
&\quad \times < \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)), dW^n(s) >\\
:=\quad & B_{211} + B_{212} + B_{213}. \quad\quad (3.62)
\end{aligned}
$$

By Ito isometry and Hölder's inequality,

$$
\begin{aligned}
&E[B_{211}]\\
\leq\quad & C\int_0^t (E[\int_{s_n^-}^s |g_n(u)\sigma^*\nabla\phi(X(u)) - g_n(s_n^-)\sigma^*\nabla\phi(X(s_n^-))|^2 du])^{\frac{1}{2}} (E[|\dot{W}^n|^2(s)])^{\frac{1}{2}} ds\\
\leq\quad & C\int_0^t (2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \quad\quad (3.63)
\end{aligned}
$$

where (3.1) and (3.2) have been used. Term B_{212} has the following bound.

$$
\begin{aligned}
&E[B_{212}]\\
\leq\quad & C\int_0^t (E[|W(s) - W(s_n^-)|^3])^{\frac{1}{3}} (E[|\dot{W}^n|^3(s)])^{\frac{1}{3}}\\
&\quad \times (E[|\sigma^*(X^n(s))(X^n(s) - X(s)) - \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-))|^3])^{\frac{1}{3}} ds\\
\leq\quad & C\int_0^t (2^n)^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} ds \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \quad\quad (3.64)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&B_{213} \\
=\ &2r \sum_k 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g_n(\frac{k-1}{2^n}) < \nabla\phi(X(\frac{k-1}{2^n})), \sigma(X(\frac{k-1}{2^n})(W(s) - W(\frac{k}{2^n})) > \\
&\times < \sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > ds \\
+\ &2r \sum_k g_n(\frac{k-1}{2^n}) < \sigma^*\nabla\phi(X(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > \\
&\times < \sigma^*(X^n(\frac{k-1}{2^n}))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > \\
:=\ &B_{2131} + B_{2132}. \quad\quad\quad (3.65)
\end{aligned}
$$

Conditioning on $\mathcal{F}_{\frac{k}{2^n}}$, it is easy to see that $E[B_{2131}] = 0$. Moreover,

$$
\begin{aligned}
&B_{2132} \\
=\ &2r \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^m (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*(X^n(\frac{k-1}{2^n})))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}))_i \\
&\times (|W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}) \\
+\ &2r \sum_k g_n(\frac{k-1}{2^n}) \sum_{i \neq j} (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*(X^n(\frac{k-1}{2^n})))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}))_j \\
&\times (W_i(\frac{k}{2^n}) - W_i(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})) \\
+\ &2r \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^m (\sigma^*\nabla\phi)_i(X(\frac{k-1}{2^n}))(\sigma^*(X^n(\frac{k-1}{2^n})))(X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}))_i(\frac{1}{2^n}) \\
:=\ &B_{21321} + B_{21322} + B_{21323}. \quad\quad\quad (3.66)
\end{aligned}
$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ and using the independence of W_i, W_j for $i \neq j$, we see that $E[B_{21321}] = 0$ and $E[B_{21322}] = 0$. Furthermore,

$$
\begin{aligned}
&E[B_{21323}] \\
=\ &2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds] \\
+\ &2rE[\int_0^t \{g_n(s_n^-) < \sigma^*\nabla\phi(X(s_n^-)), \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)) > \\
&-g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) >\}ds] \\
\leq\ &2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds] \\
&+C(\frac{1}{2^n})^{\frac{1}{2}}, \quad\quad\quad (3.67)
\end{aligned}
$$

where (3.1) and (3.2) were again used. Combining together (3.62)—(3.67) we obtain that

$$
E[B_{21}]
$$

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$$\leq 2rE[\int_0^t g_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$
$$+C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.68}$$

To bound the term B_{26}, we write it as

$$B_{26}$$
$$= 2r \int_0^t (\int_{s_n^-}^s [g_n(u) < \nabla\phi(X^n(u)), \sigma(X^n(u)dW^n(u) > -g_n(s_n^-) < \nabla\phi(X^n(s_n^-)),$$
$$\sigma(X^n(s_n^-))dW^n(u) >]) < X^n(s) - X(s), \sigma(X^n(s))dW^n(s) >$$
$$+ 2r \int_0^t (\int_{s_n^-}^s g_n(s_n^-) < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(u) >)$$
$$\times [< \sigma^*(X^n(s))(X^n(s) - X(s)) - \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)), dW^n(s) >]$$
$$+ 2r \int_0^t g_n(s_n^-) < \nabla\phi(X^n(s_n^-)), \sigma(X^n(s_n^-))(W^n(s) - W^n(s_n^-)) >$$
$$\times < \sigma^*(X^n(s_n^-))(X^n(s_n^-) - X(s_n^-)), dW^n(s) >$$
$$:= B_{261} + B_{262} + B_{263}. \tag{3.69}$$

Following the same arguments leading to the estimates for B_{211}, B_{212}, it can be shown that

$$E[B_{261}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad E[B_{261}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.70}$$

and

$$E[B_{263}]$$
$$\leq rE[\int_0^t g_n(s) < \sigma^* \nabla\phi(X^n(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$
$$+C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.71}$$

Putting together (3.57)—-(3.71) we get

$$E[B_2]$$
$$\leq rE[\int_0^t g_n(s) < \sigma^* \nabla\phi(X^n(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$
$$+2rE[\int_0^t g_n(s) < \sigma^* \nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$
$$+C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.72}$$

Now we turn to the B_3. Using the equations satisfied by X^n and X, we have

$$B_3 = 2 \int_0^t g_n(s_n^-) < \int_{s_n^-}^s \sigma(X^n(u))dW^n(u), \sigma(X^n(s))dW^n(s) >$$
$$+ 2 \int_0^t g_n(s_n^-) < L^n(s) - L^n(s_n^-), \sigma(X^n(s))dW^n(s) >$$

$$+ \quad 2\int_0^t g_n(s_n^-) < \int_{s_n^-}^s b(X^n(u))du, \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2\int_0^t g_n(s_n^-) < \int_{s_n^-}^s \sigma(X(u))dW(u), \sigma(X^n(s))dW^n(s) >$$

$$- \quad \int_0^t g_n(s_n^-) < \int_{s_n^-}^s \sigma\sigma'(X(u))du, \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2\int_0^t g_n(s_n^-) < \int_{s_n^-}^s b(X(u))du, \sigma(X^n(s))dW^n(s) >$$

$$- \quad 2\int_0^t g_n(s_n^-) < L(s) - L(s_n^-), \sigma(X^n(s))dW^n(s) >$$

$$:= \quad \sum_{i=1}^7 B_{3i}. \tag{3.73}$$

Similar to the term A_{25}, we have

$$E[B_{32}]$$
$$\leq \quad CE[\sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g_n(\frac{k-1}{2^n})(\int_{\frac{k-1}{2^n}}^s \nu(X^n(u))d|L^n|(u))2^n|W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})|ds]$$
$$\leq \quad 2CE[|L^n|(t) \sup_{|u-v|\leq\frac{1}{2^n}} (|W(u) - W(v)|)]$$
$$\leq \quad 2C(E[|L^n|^2(t)])^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}} \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.74}$$

By the same reason,

$$E[B_{37}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.75}$$

Using a similar argument as for the term A_{22}, we obtain

$$E[B_{3i}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}, i = 3, 5, 6. \tag{3.76}$$

To bound B_{31}, we write it as

$$B_{31}$$
$$= \quad 2\int_0^t g_n(s_n^-) < \int_{s_n^-}^s (\sigma(X^n(u)) - \sigma(X^n(s_n^-)))dW^n(u), \sigma(X^n(s))dW^n(s) >$$
$$+ \quad 2\int_0^t g_n(s_n^-) < \sigma(X^n(s_n^-)))(W^n(s) - W^n(s_n^-)), (\sigma(X^n(s)) - \sigma(X^n(s_n^-)))dW^n(s) >$$
$$+ \quad 2\int_0^t g_n(s_n^-) < \sigma(X^n(s_n^-))(W^n(s) - W^n(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) >$$
$$:= \quad B_{311} + B_{312} + B_{313}, \tag{3.77}$$

where

$$E[B_{311}]$$
$$\leq \quad C\int_0^t ds \int_{s_n^-}^s E[|X^n(u) - X^n(s_n^-)||\dot{W}^n(u)||\dot{W}^n(s)|]du$$
$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}}, \tag{3.78}$$

$$E[B_{312}]$$
$$\leq \quad C \int_0^t ds E[|X^n(s) - X^n(s_n^-)||W^n(s) - W^n(s_n^-)||\dot{W}^n(s)|]$$
$$\leq \quad C(\frac{1}{2^n})^{\frac{1}{2}}, \tag{3.79}$$

and

$$B_{313}$$
$$= \quad 2\sum_k 2^{2n} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g_n(\frac{k-1}{2^n})(s - \frac{k}{2^n}) ds |\sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}))|^2$$
$$+ \quad 2\sum_k g_n(\frac{k-1}{2^n}) < \sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k-1}{2^n}) - W(\frac{k-2}{2^n})),$$
$$\sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) >$$
$$:= \quad B_{3131} + B_{3132}. \tag{3.80}$$

Conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$, we see that $E[B_{3132}] = 0$. Rearranging the terms, we find that

$$B_{3131}$$
$$= \quad \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(\frac{k-1}{2^n})\{(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))^2 - \frac{1}{2^n}\}$$
$$+ \quad \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d \sum_{j \neq l}^m \sigma_{ij}(X^n(\frac{k-1}{2^n}))\sigma_{il}(X^n(\frac{k-1}{2^n}))$$
$$\times (W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))(W_l(\frac{k}{2^n}) - W_l(\frac{k-1}{2^n}))$$
$$+ \quad \int_0^t \{g_n(s_n^-) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s_n^-)) - g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))\}ds$$
$$+ \quad \int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds. \tag{3.81}$$

By conditioning and using the independence of W_j and W_l for $j \neq l$, we see that the expectation of the first two terms on the right side are zero. The expectation of the third term is bounded by $C(\frac{1}{2^n})^{\frac{1}{2}}$. Thus we have

$$E[B_{3131}]$$
$$\leq \quad E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.82}$$

As for B_{34}, we have

$$B_{34}$$

$$
\begin{aligned}
= \quad &-2\int_0^t g_n(s_n^-) < \int_{s_n^-}^s (\sigma(X(u)) - \sigma(X(s_n^-)))dW(u), \sigma(X^n(s))dW^n(s) > \\
&- 2\int_0^t g_n(s_n^-) < \sigma(X(s_n^-)))(W(s) - W(s_n^-)), (\sigma(X^n(s)) - \sigma(X^n(s_n^-)))dW^n(s) > \\
&- 2\int_0^t g_n(s_n^-) < \sigma(X(s_n^-))(W(s) - W(s_n^-)), \sigma(X^n(s_n^-))dW^n(s) > \\
:= \quad &B_{341} + B_{342} + B_{343}, \tag{3.83}
\end{aligned}
$$

Similar to the terms B_{311}, B_{312}, we have

$$
E[B_{341}] \quad \le \quad C(\frac{1}{2^n})^{\frac{1}{2}}, \tag{3.84}
$$

$$
E[B_{342}] \quad \le \quad C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.85}
$$

Moreover,

$$
\begin{aligned}
&B_{343} \\
= \quad &-2\sum_k g_n(\frac{k-1}{2^n}) < \sigma(X(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})), \\
&\qquad \sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) > \\
&- 2\sum_k 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} ds g_n(\frac{k-1}{2^n}) < \sigma(X(\frac{k-1}{2^n})(W(s) - W(\frac{k}{2^n})), \\
&\qquad \sigma(X^n(\frac{k-1}{2^n}))(W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) > \\
:= \quad &B_{3431} + B_{3432}. \tag{3.86}
\end{aligned}
$$

Conditioning on $\mathcal{F}_{\frac{k}{2^n}}$, we see that $E[B_{3432}] = 0$. Term B_{3431} can be written as follows:

$$
\begin{aligned}
&B_{3431} \\
= \quad &-2\sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X(\frac{k-1}{2^n}))\sigma_{ij}(X^n(\frac{k-1}{2^n})) \\
&\qquad \{(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))^2 - \frac{1}{2^n}\} \\
-2 \quad &\sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d \sum_{j\neq l}^m \sigma_{ij}(X(\frac{k-1}{2^n}))\sigma_{il}(X^n(\frac{k-1}{2^n})) \\
&\qquad \times (W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n}))(W_l(\frac{k}{2^n}) - W_l(\frac{k-1}{2^n})) \\
-2 \quad &\int_0^t \{g_n(s_n^-) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X(s_n^-))\sigma_{ij}^2(X^n(s_n^-)) - g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X(s))\sigma_{ij}^2(X^n(s))\}ds \\
-2 \quad &\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}(X(s))\sigma_{ij}(X^n(s))ds. \tag{3.87}
\end{aligned}
$$

By conditioning and using the independence of W_j and W_l for $j \neq l$, it follows that the expectation of the first two terms on the right side are zero and the expectation of the third term is bounded by $C(\frac{1}{2^n})^{\frac{1}{2}}$. Thus we have

$$
E[B_{3431}]
$$
$$
\leq E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds] + C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (3.88)
$$

It follows from (3.73) —(3.88) that

$$
E[B_3]
$$
$$
\leq E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds] + C(\frac{1}{2^n})^{\frac{1}{2}}
$$
$$
- 2E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}(X(s))\sigma_{ij}(X^n(s))ds]. \qquad (3.89)
$$

To bound B_4, denote by $\nabla \sigma = (\nabla \sigma_{ij}) \in R^{d \times m} \otimes R^d$ and $\sigma^* \nabla \sigma \in R^{d \times m} \otimes R^d$ the linear mappings defined by

$$
< \nabla \sigma, y >= (< \nabla \sigma_{ij}, y >) \in R^{d \times m}, \quad y \in R^d,
$$

$$
< \sigma^* \nabla \sigma, x >= (< \sigma^* \nabla \sigma_{ij}, x >) \in R^{d \times m}, \quad x \in R^m.
$$

Observe that

$$
\begin{aligned}
B_4 &= 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), (\int_{s_n^-}^s < \nabla\sigma(X^n(u)), \sigma(X^n(u))dW^n(u) >)dW^n(s) > \\
&+ 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), (\int_{s_n^-}^s < \nabla\sigma(X^n(u)), \nu(X^n(u))d|L^n|_u >)dW^n(s) > \\
&+ 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), (\int_{s_n^-}^s < \nabla\sigma(X^n(u)), b(X^n(u))du >)dW^n(s) > \\
&:= B_{41} + B_{42} + B_{43}. \qquad (3.90)
\end{aligned}
$$

As other similar terms treated above, we can show that

$$
E[B_{43}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (3.91)
$$

$$
E[B_{42}] \leq CE[|L^n|_t \sup_{|u-v| \leq \frac{1}{2^n}} |W(u) - W(v)|] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \qquad (3.92)
$$

B_{41} can be further split as

$$
\begin{aligned}
B_{41} &= 2\int_0^t g_n(s_n^-) < X^n(s_n^-) - X(s_n^-), \\
& \quad (\int_{s_n^-}^s < \sigma^* \nabla\sigma(X^n(u)) - \sigma^* \nabla\sigma(X^n(s_n^-)), dW^n(u) >)dW^n(s) >
\end{aligned}
$$

$$+ \quad 2\sum_k g_n(\frac{k-1}{2^n}) < X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}),$$

$$< \sigma^* \nabla \sigma(X^n(\frac{k-1}{2^n})), W(\frac{k-1}{2^n}) - W(\frac{k-2}{2^n}) > (W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) >$$

$$+ \quad 2\sum_k 2^{2n} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (s - \frac{k}{2^n}) ds g_n(\frac{k-1}{2^n}) < X^n(\frac{k-1}{2^n}) - X(\frac{k-1}{2^n}),$$

$$< \sigma^* \nabla \sigma(X^n(\frac{k-1}{2^n})), W(\frac{k}{2^n}) - W(\frac{k-1}{2^n}) > (W(\frac{k}{2^n}) - W(\frac{k-1}{2^n})) >$$

$$:= \quad B_{411} + B_{412} + B_{413}. \tag{3.93}$$

As other similar terms above, we have

$$E[B_{411}] \le C(\frac{1}{2^n})^{\frac{1}{2}}, \qquad E[B_{412}] = 0. \tag{3.94}$$

On the other hand, we have

$$
\begin{aligned}
B_{413} \quad = \quad & \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d (X_i^n(\frac{k-1}{2^n}) - X_i(\frac{k-1}{2^n})) \\
& \times \sum_{\substack{j \ne l}}^m (\sigma^* \nabla \sigma_{ij})_l(X^n(\frac{k-1}{2^n}))(W_l(\frac{k}{2^n}) - W_l(\frac{k-1}{2^n}))(W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})) \\
+ \quad & \sum_k g_n(\frac{k-1}{2^n}) \sum_{i=1}^d (X_i^n(\frac{k-1}{2^n}) - X_i(\frac{k-1}{2^n})) \\
& \times \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j(X^n(\frac{k-1}{2^n}))\{|W_j(\frac{k}{2^n}) - W_j(\frac{k-1}{2^n})|^2 - \frac{1}{2^n}\} \\
+ \quad & \int_0^t \{g_n(s_n^-) \sum_{i=1}^d (X_i^n(s_n^-) - X_i(s_n^-)) \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j(X^n(s_n^-)) \\
& -g_n(s) \sum_{i=1}^d (X_i^n(s) - X_i(s)) \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j(X^n(s))\}ds \\
+ \quad & \int_0^t g_n(s) \sum_{i=1}^d (X_i^n(s) - X_i(s)) \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j(X^n(s))ds. \tag{3.95}
\end{aligned}
$$

Using the independence of W_j and W_l for $j \ne l$, (3.1) and (3.2), by conditioning on $\mathcal{F}_{\frac{k-1}{2^n}}$ we obtain from (3.95) that

$$
\begin{aligned}
& E[B_{413}] \\
\le \quad & E[\int_0^t g_n(s) \sum_{i=1}^d (X_i^n(s) - X_i(s)) \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j(X^n(s))\}ds] \\
& + C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.96}
\end{aligned}
$$

Combining (3.90)—(3.96) yields that

$$E[B_4]$$

$$\leq \quad E[\int_0^t g_n(s) \sum_{i=1}^d (X_i^n(s) - X_i(s)) \sum_{j=1}^m (\sigma^* \nabla \sigma_{ij})_j (X^n(s))\} ds]$$

$$+ C(\frac{1}{2^n})^{\frac{1}{2}}. \tag{3.97}$$

The lemma is proved by putting together (3.56), (3.72), (3.89) and (3.99).

Proof of Theorem 2.2: (Continued). Choose $r < -\frac{2C_0}{\alpha}$, where α, C_0 are the constants appeared in the assumptions (D.1) and (D.2). By the Lipschitz continuity of the coefficients and boundedness of ϕ, ϕ'', $\nabla \phi$, $\sigma \sigma'$ on the domain $\bar{D}$, it follows from (3.10) that

$$E[f_n(t)]$$

$$\leq \quad C_r E[\int_0^t f_n(s) ds]$$

$$+ \quad E[\int_0^t \{< r f_n(s) \nabla \phi(X(s)) - 2g_n(s)(X^n(s) - X(s)), \nu(X(s)) >\} d|L|(s)]$$

$$+ \quad r E[\int_0^t f_n(s) < \nabla \phi(X^n(s)), \sigma(X^n(s)) dW^n(s) >]$$

$$+ \quad E[\int_0^t \{< r f_n(s) \nabla \phi(X^n(s)) + 2g_n(s)(X^n(s) - X(s)), \nu(X^n(s)) >\} d|L^n|(s)]$$

$$+ \quad 2 E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma(X^n(s)) dW^n(s) >]$$

$$- \quad E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma \sigma'(X(s)) > ds]$$

$$+ \quad E[\int_0^t g_n(s) tr(\sigma \sigma^*(X(s))) ds]$$

$$- \quad 2r E[\int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^* \nabla \phi(X(s)) > ds]. \tag{3.98}$$

In view of $r < 0$ and the assumptions (D.1) and (D.2), we deduce that

$$< r f_n(s) \nabla \phi(X(s)) - 2g_n(s)(X^n(s) - X(s)), \nu(X(s)) >$$
$$= \quad g_n(s)[r < \nabla \phi(X(s)), \nu(X(s)) > |x^n(s) - X(s)|^2 - 2 < X^n(s) - X(s), \nu(X(s)) >]$$
$$\leq \quad g_n(s)[r\alpha |x^n(s) - X(s)|^2 + 2C_0 |X^n(s) - X(s)|^2] \leq 0, \tag{3.99}$$

and similarly

$$< r f_n(s) \nabla \phi(X^n(s)) + 2g_n(s)(X^n(s) - X(s)), \nu(X^n(s)) >$$
$$\leq \quad 0. \tag{3.100}$$

Thus, using Lemma 3.1 and Lemma 3.2, taking into account (3.99) and (3.100) we obtain from (3.98) that

$$E[f_n(t)]$$

$$\leq \quad C_r E[\int_0^t f_n(s) ds] + C(\frac{1}{2^n})^{\frac{1}{2}}$$

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$$-\quad E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X(s)) > ds]$$

$$+\quad E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X(s))ds]$$

$$-\quad 2rE[\int_0^t g_n(s) < \sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X(s)) > ds]$$

$$+\quad 2r\int_0^t < g_n(s)\sigma^*(X^n(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds$$

$$-\quad 2r\int_0^t < g_n(s)\sigma^*(X(s))(X^n(s) - X(s)), \sigma^*\nabla\phi(X^n(s)) > ds$$

$$+\quad 2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)), \sigma^*(X^n(s))(X^n(s) - X(s)) > ds]$$

$$+\quad E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2(X^n(s))ds]$$

$$+\quad E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X^n(s)) > ds]$$

$$-2\quad E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}(X(s)\sigma_{ij}(X^n(s))ds]$$

$$\leq\quad C_r E[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}}$$

$$+\quad E[\int_0^t g_n(s) \sum_{i=1}^d \sum_{j=1}^m (\sigma_{ij}(X(s)) - \sigma_{ij}(X^n(s)))^2 ds]$$

$$+\quad 2r\int_0^t < g_n(s)(\sigma^*(X^n(s)) - \sigma^*(X(s)))(X^n(s) - X(s)),$$
$$\sigma^*\nabla\phi(X^n(s)) > ds$$

$$+\quad 2rE[\int_0^t g_n(s) < \sigma^*\nabla\phi(X(s)),$$
$$(\sigma^*(X^n(s)) - \sigma^*(X(s)))(X^n(s) - X(s)) > ds]$$

$$+\quad E[\int_0^t g_n(s) < X^n(s) - X(s), \sigma\sigma'(X^n(s)) - \sigma\sigma'(X(s)) > ds]$$

$$\leq\quad CE[\int_0^t f_n(s)ds] + C(\frac{1}{2^n})^{\frac{1}{2}}, \tag{3.101}$$

where the Lipschitz continuity of the coefficients and the fact that $f_n(s) = g_n(s)|X^n(s) - X(s)|^2$ have been used. Finally by the Gronwall's inequality, we obtain

$$E[f_n(t)] \leq C(\frac{1}{2^n})^{\frac{1}{2}} \to 0 \tag{3.102}$$

as $n \to \infty$, completing the proof of (3.9), hence the theorem.

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